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Application of complex canonical point transformations to linear second-order differential equations

M C Spinel

Departamento de Física, Universidad Nacional de Colombia, Santafé de Bogotá, Colombia

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Abstract. We present a canonical method to solve one-dimensional linear differential equations making use of pseudodifferential calculus. We apply two successive canonical point transformations on the cartesian momentum and position spaces to obtain a nonlinear complex-valued canonical transformation which maps a very simple linear differential equation into the desired differential equation. This method yields a closed contour integral representation for the exact solution in terms of arbitrary functions, which may be determined from the mapping equations in a similar way to that followed in classical mechanics. This method does not require the completeness condition on the intermediary states and avoids calculation of the kernel of the generator. We explicitly develop the case of second-order differential equations and give some standard examples to show how this method works.

1. Introduction

The advantage of classical canonical methods to solve Hamilton equations is well known. These methods seek for the transformation which maps a given Hamiltonian system onto a simpler one whose solution is known or easier to determine. The desired solution may be obtained applying the inverse transformation to the image solution [1].

Applications of these canonical methods in quantum mechanics assume unitary canonical transformations in order to assure the completeness condition for the states of the image position and momentum operators. These unitary transformation allow one to expand the states of a Hamiltonian operator in terms of the states of a simpler one [2]. Semiclassical approximation methods commonly give an approach to the Schwartz kernel (or matrix elements in coordinate representation) of unitary generators [3]. Complex-valued extensions for linear transformations have been studied by Kramer *et al.* [4] and for classical canonical generators of nonlinear transformations, including criteria of quantum exactness, by Jung and Krüger [5]. Representations in quantum mechanics of nonlinear and non-bijective canonical transformations have been discussed by Moshinsky and Seligman [6] and the unitary representation for sequences of real linear and point transformations has been discussed by Leyvraz and Seligman [7]. However, in these treatments the calculation of the kernel for the integral form of the generator is always invoked.

By application of pseudodifferential calculus we present a canonical method to solve one-dimensional linear differential equations which does not require the

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unitarity condition on the generator and avoids the calculation of its Schwartz kernel. We apply two successive canonical point transformations on the Cartesian momentum and position spaces to obtain a nonlinear complex-valued canonical transformation which maps a very simple linear equation into the desired differential equation. This method yields a closed contour integral representation for the exact solution in terms of arbitrary complex-valued canonical transformation which maps a very simple linear equation into the desired differential equation. This method yields a closed contour integral representation for the exact solution in terms of arbitrary complex-valued functions which can be determined from the mapping equations in a similar way to that followed in classical mechanics.

In section 2 we define notation and present the basic relations for normal ordered pseudodifferential operators (OPO) which establish a correspondence between an OPO and a well-behaved function defined on the classical phase space, called the symbol of the operator. In section 3 we study the complex extension of canonical point transformations. After studying the solution of the canonicity condition on the image position and momentum operators we determine the normal symbol of the generator and the corresponding mapping equations. In these sections we restrict ourselves to the case of one degree of freedom because we are interested in one-dimensional differential equations, however, generalization to more degrees of freedom is straightforwardly obtained.

The description of the method in an operational way is given in section 4 where the complex extension for the solution in terms of the normal symbols of the generators is included. Then, by the selection of a very simple initial equation in section 5, we construct a closed contour integral representation for the solution and explicitly describe how to obtain it in the case of second-order differential equations. Some standard examples which illustrate how this method works are given in section 6.

2. Preliminary relations

In this section we define notation and recall the relations that we will require concerning coordinate representation in quantum mechanics and pseudodifferential calculus.

The Hermitian position and momentum operators $\hat{q}$ and $\hat{p}$ satisfy the commutation relations $[\hat{q},\hat{q}]=[\hat{p},\hat{p}]=0$ and

$$[\hat{q},\hat{p}]=\hat{q}\hat{p}-\hat{p}\hat{q}=\mathrm{i}\,\hbar\hat{1} \tag{1}$$

where $\hat{1}$ is the identity operator. $|q\rangle$ and $|p\rangle$ denote the eigenstates of $\hat{q}$ and $\hat{p}$ corresponding to the eigenvalues $q\in\mathfrak{R}_q$ (Cartesian coordinate space) and $p\in\mathfrak{R}_p$ (momentum space), respectively. The coordinate representation of $\hat{q}$ and $\hat{p}$ is given by

$$\begin{aligned} &\hat{q}|q\rangle=q|q\rangle &&\langle q|\hat{q}=q\langle q| \\ &\hat{p}|q\rangle=\mathrm{i}\hbar\partial_q|q\rangle &&\langle q|\hat{p}=-\mathrm{i}\hbar\partial_q\langle q| \end{aligned} \tag{2}$$

where ∂_q is the gradient operator acting on $\mathfrak{R}_q$ and $\langle q|$ is the adjoint of $|q\rangle$: $\langle q|=|q\rangle^{\dagger}$.

The states $|q\rangle$, like $|p\rangle$, are assumed to be orthogonal:

$$\langle q|x\rangle=\delta(q-x)=(2\pi\hbar)^{-1}\int \mathrm{d}y\,\exp\left\{\frac{\mathrm{i}}{\hbar}(q-x)y\right\}$$

where $\delta(q-x)$ is the delta Dirac distribution.

Let $|u\rangle$ be an arbitrary state and $\hat{A}$ be a polynomial operator in $\hat{p}$ whose coefficients, which are functions of $\hat{q}$, are to the left of the momentum operators. By application of (2) the coordinate representation of $\hat{A}|u\rangle$ leads to the following differential equation:

$$\langle q|\hat{A}|u\rangle = A(x, -i\hbar\partial_q)u(q)|_{x=q} \tag{3}$$

where $\langle q|u\rangle = u(q)$ is the coordinate representation of $|u\rangle$.

On the other hand, standard definitions of ordered pseudodifferential operators (OPO) establish a one-to-one correspondence between an OPO and a well-behaved function defined on the classical phase space $\mathcal{R}_q \times \mathcal{R}_p$ [8]. This function is called the symbol of the operator.

The definition of a normal (or right) OPO (see section 4 in [9]) is

$$A(x, \mathcal{D})u(x) = (2\pi\hbar)^{-1} \int\int \mathrm{d}y\, \mathrm{d}\xi A(x, \xi) \exp\left\{\frac{\mathrm{i}}{\hbar}(x-y)\xi\right\} u(y) \tag{4}$$

where $\mathcal{D} = -i\hbar\partial_x$, $u(x)$ belongs to the Schwartz space $\mathcal{S}(\mathcal{R})$ and $A(x, \xi) \in \mathcal{S}(\mathcal{R}^2)$. $A(x, \xi)$ is called the normal (or right) symbol of the pseudodifferential operator $A(x, \mathcal{D})$ because if $A(x, \xi)$ is a polynomial or a convergent series in ξ, the operator is obtained from the symbol replacing ξ by $\mathcal{D}$ to the right of the coefficients and hence the left-hand side of this equation leads to the normal coordinate representation (3). In such a case an equivalent definition for a normal OPO, developed by Krüger [10], is given by

$$\hat{A} = A(\partial_q, \partial_p) \exp(q\hat{q}) \exp(p\hat{p})|_{q=0, p=0}$$

or

$$\hat{A} = A(\hat{q}, \partial_p) \exp(p\hat{p})|_{p=0} \tag{5}$$

where $A(q, p)$, the normal symbol of $\hat{A}$, is obtained from the differential operator $A(\partial_q, \partial_p)$ replacing ∂_q by q and ∂_p by p. To show the equivalence we consider the coordinate representation of $\hat{A}|u\rangle$:

$$\langle x|\hat{A}|u\rangle = A(x, \partial_p) \exp(-i\hbar p\partial_x)u(x)|_{p=0}$$

$$\langle x|\hat{A}|u\rangle = \int \mathrm{d}\xi A(x, \xi)\delta(\partial_p - \xi) \exp(-i\hbar p\partial_x)u(x)|_{p=0}.$$

Replacement of the delta distribution by the integral form given above yields

$$\langle x|\hat{A}|u\rangle = (2\pi\hbar)^{-1} \int\int \mathrm{d}\xi\, \mathrm{d}v A(x, \xi) \exp\left\{-\frac{\mathrm{i}}{\hbar}\xi v\right\} u(x+v)$$

and hence the change of variable $y = x + v$ leads to (4).

The normal symbol of a product of operators $\hat{A}\hat{B}$ can be obtained from (5) as follows:

$$\hat{A}\hat{B}=A(\hat{q},\partial_p)\exp(p\hat{p})B(\hat{q},\partial_y)\exp(y\hat{p})|_{p=0,y=0}.$$

With $\exp(p\hat{p})B(\hat{q},\partial_y)=B(\hat{q}-\mathrm{i}\hbar p,\partial_y)\exp(p\hat{p})$ and the identity

$$A(\partial_q,\partial_p)f(q,p)\exp(q\hat{q})\exp(p\hat{p})|_{q=0,p=0}$$
$$=\{f(\partial_\alpha,\partial_\beta)A(\alpha,\beta)|_{\alpha=\partial_q,\beta=\partial_p}\}\exp(q\hat{q})\exp(p\hat{p})|_{q=0,p=0} \tag{6}$$

where $f(q,p)$ is an analytic function, we obtain

$$\hat{A}\hat{B}=\{B(\hat{q}-\mathrm{i}\hbar\partial_\beta,\alpha)A(\hat{q},\beta)|_{\alpha=\beta,\beta=\partial_p}\}\exp(p\hat{p})|_{p=0}.$$

Comparison with

$$\hat{A}\hat{B}\equiv(A\circ B)(\hat{q},\partial_p)\exp(p\hat{p})|_{p=0}$$

where $(A\circ B)(q,p)$ denotes the normal symbol of $\hat{A}\hat{B}$, leads to

$$(A\circ B)(q,p)=B(q-\mathrm{i}\hbar\partial_p,y)A(q,p)|_{y=p}=A(x,p-\mathrm{i}\hbar\partial_q)B(q,p)|_{x=q}. \tag{7}$$

We call normal representation the pseudodifferential calculus developed from the definition of normal OPO, and this is the representation that we will use in the next sections. However, to study the Hermiticity of the image position and momentum operators under canonical point transformations is more adequate than the Weyl representation because of its symmetric definition. Therefore to end this section we obtain the relation between the normal and Weyl symbols for a given operator.

With a similar procedure to that given for definition (5), it can be shown that Krüger's definition for the Weyl ordering prescription [10]:

$$\hat{A}=A_w(\partial_p,\partial_p)\exp(q\hat{q}+p\hat{p})|_{q=0,p=0},$$

where $A_w(q,p)$ denotes the Weyl symbol of $\hat{A}$, is equivalent to the standard definition (for the Weyl integral definition see, for example, equation (4.1) in [9]). The Hermiticity of $\exp(q\hat{q}+p\hat{p})$ shows that $[A_w(q,p)]^*$ corresponds to $\hat{A}^\dagger$ and hence that real Weyl symbols correspond to Hermitian operators, and conversely. Applying the identity $\exp(q\hat{q}+p\hat{p})=\exp(-\frac{1}{2}\mathrm{i}\hbar qp)\exp(q\hat{q})\exp(p\hat{p})$ and (6) with $f(q,p)=\exp(-\frac{1}{2}\mathrm{i}\hbar qp)$, we obtain

$$\hat{A}=\{\exp(-\tfrac{1}{2}\mathrm{i}\hbar\partial_\alpha\partial_\beta)A_w(\alpha,\beta)|_{\alpha=\partial_q,\beta=\partial_p}\}\exp(p\hat{p})|_{p=0}$$

and comparison with (5) yields the desired relation:

$$A(q,p)=\exp(-\tfrac{1}{2}\mathrm{i}\hbar\partial_p\partial_p)A_w(q,p). \tag{8}$$

3. Complex canonical point transformations

In this section we obtain the normal symbols for the transformation operator (or generator) and the image position and momentum operators for canonical point transformations on the Cartesian position and momentum spaces $\mathfrak{R}_q$ and $\mathfrak{R}_p$. These transformations will be called q-point CT and p-point CT, respectively.

Let $\hat{T}$ be the operator of a q-point CT which maps the Hermitian position and momentum operators $\hat{q}$ and $\hat{p}$ to the image operators $\hat{Q}=f(\hat{q})$ and $\hat{P}$ according to

$$\hat{T}\hat{q}=f(\hat{q})\hat{T} \qquad \hat{T}\hat{p}=\hat{P}\hat{T}$$

where $f(x) \in \mathscr{C}^\infty(\mathscr{R}_q)$. The image operators must satisfy the same commutators as the original ones, i.e. $[\hat{Q}, \hat{Q}] = [\hat{P}, \hat{P}] = 0$ and

$$[\hat{Q}, \hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar\hat{1} \tag{9}$$

or explicitly $f(\hat{q})\hat{P} - \hat{P}f(\hat{q}) = i\hbar\hat{1}$. Equation (9) will be referred to as the canonicity condition.

By application of (7), the normal representation of these equations yields the following system of differential equations for the normal symbols of $\hat{T}$ and $\hat{P}$:

$$[(q - i\hbar\partial_p) - f(q)]T(q, p) = 0 \tag{10}$$

$$[p - P(x, p - i\hbar\partial_q)]T(q, p)|_{x=q} = 0 \tag{11}$$

$$[f(q) - f(q - i\hbar\partial_p)]P(q, p) = i\hbar. \tag{12}$$

A particular solution of (12) is

$$P_0(q, p) = \frac{1}{f'(q)} p + r(q) \tag{13}$$

for $f'(q) = \partial_q f(q) \neq 0$, while for the homogeneous part one obtains the following general solution:

$$P_h(q, p) = \sum_k b_k(q) \exp\left(\frac{i}{\hbar} p\nu_k(q)\right)$$

up to an additive function of q which may be absorbed by $r(q)$ in (13). Here $b_k(q)$, $\nu_k(q) \in \mathscr{C}^\infty(\mathscr{R})$ and $\nu_k(q)$ is one of the multiple solutions of the functional equation $[f(q) - f(q + \nu(q))] = 0$. So

$$P(q, p) = \frac{1}{f'(q)} p + r(q) + \sum_k b_k(q) \exp\left(\frac{i}{\hbar} p\nu_k(q)\right)$$

is the most general solution of the canonicity condition for q-point CT.

For selecting solutions regular in $\hbar$ at $\hbar = 0$, we analyse the $\hbar$-dependence of $P(q, p)$ and write the last equation in the form

$$P(q, p) = \mathscr{P}_0(q, p) + \mathscr{P}_1(q, p) \tag{14}$$

where

$$\mathscr{P}_0(q, p) = \frac{1}{f'(q)} p + s(q) \qquad \mathscr{P}_1(q, p) = \sum_k b_k(q) \left\{\exp\left(\frac{i}{\hbar} p v_k(q)\right) - 1\right\}$$

with $s(q) = r(q) + \Sigma_l b_l(q)$. Here $\mathscr{P}_0$ admits an expansion in powers series of $\hbar$:

$$\mathscr{P}_0(q, p; \hbar) = \sum_{k=0}^{\infty} \hbar^k \mathscr{P}_{ok}(q, p)$$

while $\mathscr{P}_1$ admits an expansion of $\hbar^{-1}$:

$$\mathscr{P}_1(q, p; \hbar^{-1}) = \sum_{k=1}^{\infty} \hbar^{-k} \mathscr{P}_{1k}(q, p)$$

which is not regular at $\hbar=0$.

Hence discarding $\mathcal{P}_1$ from (14) we obtain the regular solution

$$P(q,p)=\frac{1}{f'(q)}p+s(q). \tag{15}$$

This equation together with $Q(q,p)=f(q)$ are the mapping equations of a canonical point transformation on $\mathcal{R}_q$ in classical mechanics [1].

To study the existence of $\hat{T}$ for a given $f(q)$ we recall equations (10) and (11). Integration of (10) yields

$$T(q,p)=A(q)\exp\left\{\frac{\mathrm{i}}{\hbar}[f(q)-q]p\right\} \tag{16}$$

where $A(q)\in\mathcal{C}^\infty(\mathcal{R})$. With (15) and (16), equation (11) becomes

$$s(q)=\frac{1}{f'(q)}\,\mathrm{i}\hbar\partial_q\ln A(q)$$

and hence

$$P(q,p)=\frac{1}{f'(q)}\{p+\mathrm{i}\hbar\partial_q\ln A(q)\}. \tag{17}$$

The last equation together with

$$Q(q,p)=f(q) \tag{18}$$

define the mapping equations for q-point CT on the phase space. The complex-valued function $A(q)$ can be determined up to a constant factor from the knowledge of $P(q,p)$ for a given $f(q)$.

In order to check the Hermiticity of the image operators $\hat{Q}$ and $\hat{P}$ we analyse their Weyl symbols. By application of (8) to the normal symbols (17) and (18) we obtain

$$P_{\mathrm{w}}(q,p)=\frac{1}{f'(q)}\{p+\tfrac{1}{2}\,\mathrm{i}\hbar\partial_q\ln[A^2(q)/f'(q)]\}$$

$$Q_{\mathrm{w}}(q,p)=f(q).$$

Since real Weyl symbols correspond to Hermitian operators we infer that even for $f(q)$ real, when $\hat{Q}$ is Hermitic, $\hat{P}$ will in general not be Hermitic unless $A^2(q)/f'(q) = C$, where C is a constant. This analysis shows that (16), (17) and (18) define in general a complex-valued q-point CT even for $f(q)$ real if $A^2(q)/f'(q) \neq C$.

Let now $\hat{U}$ be the operator of a p-point CT defined by

$$\hat{U}\hat{q} = \hat{Q}\hat{U} \qquad \hat{U}\hat{p} = g(\hat{p})\hat{U}$$

where $g(x) \in \mathscr{C}^\infty(\mathscr{R}_p)$, with the canonicity condition

$$\hat{Q}g(\hat{p}) - g(\hat{p})\hat{Q} = \mathrm{i}\hbar\hat{1}.$$

Normal representation of these equations yields

$$[p - g(p - \mathrm{i}\hbar\partial_q)]U(q,p) = 0 \tag{19}$$

$$[(q - \mathrm{i}\hbar\partial_p) - Q(x, p - \mathrm{i}\hbar\partial_q)]U(q,p)|_{x=q} = 0 \tag{20}$$

$$[g(p) - g(p - \mathrm{i}\hbar\partial_q)]Q(q,p) = \mathrm{i}\hbar$$

In analogy with (12) a solution of the last equation regular in $\hbar$ at $\hbar = 0$ is

$$Q(q,p) = \frac{1}{g'(p)}q + r(p)$$

for $g'(p) = \partial_p g(p) \neq 0$.

A general solution of (19) is

$$U(q,p) = B(p)\exp\left\{\frac{\mathrm{i}}{\hbar}[g^{-1}(p) - p]q\right\} \tag{21}$$

where $B(p) \in \mathscr{C}^\infty(\mathscr{R})$. With these solutions (20) becomes

$$r(p) = -\mathrm{i}\hbar\frac{1}{g'(p)}\partial_p \ln B(g(p))$$

and hence

$$Q(q,p) = \frac{1}{g'(p)}\{q - \mathrm{i}\hbar\partial_p \ln B(g(p))\} \tag{22}$$

which together with

$$P(q,p) = g(p) \tag{23}$$

define the mapping equations for p-point CT. As in the q-point CT case, for a given $g(p)$ the complex-valued function $B(p)$ can be determined up to a constant factor from the knowledge of $Q(q,p)$.

The corresponding Hermiticity test for $\hat{Q}$ and $\hat{P}$ shows that (21), (22) and (23) define complex-valued p-point CT even for $g(p)$ real if $g'(p)B^2(g(p)) \neq \text{constant}$.

4. General method

Let $\hat{\Omega}_0|\varphi_0\rangle = 0$ represent the initial differential equation where $|\varphi_0\rangle$ is assumed known. By means of a p-point CT satisfying $\hat{U}\hat{\Omega}_0 = \hat{\Omega}_1\hat{U}$ the initial equation is mapped into the intermediary equation $\hat{\Omega}_1|\varphi_1\rangle = 0$, where

$$|\varphi_1\rangle = \hat{U}|\varphi_0\rangle. \tag{24}$$

Following with a q-point CT defined by $\hat{T}\hat{\Omega}_1 = \hat{\Omega}\hat{T}$ the intermediary equation can be mapped into the linear differential equation we want to solve: $\hat{\Omega}|\psi\rangle = 0$, where

$$|\psi\rangle = \hat{T}|\varphi_1\rangle. \tag{25}$$

Hence $|\psi\rangle = \hat{T}\hat{U}|\varphi_0\rangle$ is obtained if the generators $\hat{T}$ and $\hat{U}$ can be determined. We can think of $|\varphi_0\rangle$, $|\varphi_1\rangle$ and $|\psi\rangle$ as the eigenstates of the operators $\hat{\Omega}_0$, $\hat{\Omega}_1$ and $\hat{\Omega}$ corresponding to the eigenvalue zero so that the canonical transformations $\hat{U}$ and $\hat{T}$ map the states of $\hat{\Omega}_0$ and $\hat{\Omega}_1$ into states of $\hat{\Omega}_1$ and $\hat{\Omega}$, respectively, all of them corresponding to the same eigenvalue. These operators are in general non-Hermitian.

Applying definition (5), equation (24) becomes

$$|\varphi_1\rangle = U(\hat{q}, \partial_p)\exp(p\hat{p})|\varphi_0\rangle|_{p=0}$$

whose coordinate representation, called the wavefunction, is

$$\varphi_1(q) = (\hat{U}\varphi_0)(q) \equiv U(q, \partial_p)\exp(-i\hbar p\partial_q)\varphi_0(q)|_{p=0}.$$

This definition, like calculations which led to the complex extension of p-point and q-point CT, makes no assumption of integral representations, delta Dirac distributions, Fourier transformations or completeness condition on the intermediary states $|\varphi_0\rangle$ and $|\varphi_1\rangle$. So, for analytic functions, the last equation can be directly complexified through the following definition:

$$\varphi_1(z) = (\hat{U}\varphi_0)(z) \equiv U(z, \partial_v)\exp(-i\hbar v\partial_z)\varphi_0(z)|_{v=0}$$

with $z, v \in \mathscr{C}$, $\varphi_0(z)$ an analytic function and $U(z, v)$ the complex extension of the analytic normal symbol of $\hat{U}$ which, according to (5), is defined by

$$\hat{U} = U(\partial_z, \partial_v)\exp(z\hat{q})\exp(v\hat{p})|_{z=0,\, v=0}.$$

Recalling the normal symbol (21) one obtains for the solution of the intermediary equation the general expression

$$\varphi_1(z) = \left\{ B(-i\hbar\partial_z)\exp\left\{\frac{i}{\hbar}[\alpha - g(\alpha)]z'\right\}\Big|_{\alpha=g^{-1}(-i\hbar\partial_z)} \right\}\varphi_0(z)|_{z'=z}. \tag{26}$$

Similarly, with (16) the equation (25) yields

$$\psi(z) = A(z)\exp\left\{\frac{i}{\hbar}[f(z) - z]\partial_v\right\}\exp(-i\hbar v\partial_z)\varphi_1(z)|_{v=0}$$

and hence

$$\psi(z) = A(z)\varphi_1(f(z)). \tag{27}$$

From this equation one notes that the Miller–Good transformation $\varphi(x) \mapsto \psi(x) = ((\partial_x S)x))^{-1/2}\varphi(S(x))$, with $x \in \mathfrak{R}$, $s \in \mathscr{C}$ [11], is precisely a particular q-point CT with $f(x) = S(x)$ and $A(x) = (\partial_x f(x))^{-1/2}$.

Applying the same complex extension to the initial and image differential equations we obtain

$$(\hat{\Omega}_0\varphi_0)(z) = \Omega_0(z', -i\hbar\partial_z)\varphi_0(z)|_{z'=z} = 0 \tag{28}$$

$$\Omega_1(z', -i\hbar\partial_z)\varphi_1(z)|_{z'=z} = 0 \tag{29}$$

$$\Omega(z', -i\hbar\partial_z)\psi(z)|_{z'=z} = 0. \tag{30}$$

In particular for $\hat{\Omega} = \hat{H} - \alpha$, where $\hat{H}$ is a Hamiltonian operator, (30) yields for z real the standard coordinate representation of a time-independent Schrödinger equation

$$H(x', -i\hbar\partial_x)\psi(x)|_{x'=x} = \alpha\psi(x)$$

whose solution is then given by (27) with $z=x$. If $\hat{\Omega}_0=\hat{H}_0-\alpha_0$ and $\hat{\Omega}_1=\hat{H}_1-\alpha_1$, the spectrum is mapped according to the sequence $\{\alpha_0\}\mapsto\{\alpha_1\}\mapsto\{\alpha\}$.

It is worth noting the very simple expression (27) for the image solution under q-point CT. In particular, when one is interested in determining the states for a given Hamiltonian $\hat{H}=(2m)^{-1}\hat{p}^2+V(\hat{q})$ from a knowledge of the states of a simpler one $\hat{H}_0=(2m)^{-1}\hat{p}^2+V(\hat{q})$ from a knowledge of the states of a simpler one $\hat{H}_0=(2m)^{-1}\hat{p}^2+V_0(\hat{q})$, it is possible that only a q-point CT may be required and hence the exact $\psi(z)$ is directly given by (27) with $\varphi_1(z)=\varphi_0(z)$. When a composite transformation is required the success of this method to give exact solutions depends on the adequate selection of $\hat{\Omega}_0$ (and hence of φ_0) so that equation (26) can be calculated exactly.

The functions f, A, g, and B defining the solutions may be obtained from the mapping equations (17), (18), (22) and (23) in such a way that they map the normal symbols of $\hat{\Omega}_0$ and $\hat{\Omega}_1$ into the normal symbols of $\hat{\Omega}_1$ and $\hat{\Omega}$, respectively. In the next section we explicitly describe this procedure for the case of second-order differential equations.

5. Closed contour integral representation

Choosing for the initial equation the following simple form

$$\hat{\Omega}_0|\varphi_0\rangle\equiv\hat{q}|\varphi_0\rangle=0$$

equation (28) yields

$$z\varphi_0(z)=0. \tag{31}$$

A solution of this equation defined for all values of z is given by the closed contour integral

$$\varphi_0(z)=\oint_\gamma \mathrm{d}w\exp\left\{\frac{\mathrm{i}}{\hbar}wz\right\}, \tag{32}$$

where γ is a closed path of integration around the point $w=w_0$, for any w_0 in the extended complex plane (Riemann sphere) which is formed by compactification with the point ∞. Indeed, when $w_0\in\mathscr{C}$ the path can be defined by $w(t)=w_0+r\exp(\mathrm{i}t)$, $r>0$, $-\pi\leqslant t<\pi$, so that the left-hand side of (31) becomes

$$z\varphi_0(z)=\hbar\exp\left(\frac{\mathrm{i}}{\hbar}w_0z\right)\int_{-\pi}^{\pi}\mathrm{d}t\left[\frac{\mathrm{i}zr}{\hbar}\exp(\mathrm{i}t)\right]\exp\left(\frac{\mathrm{i}zr}{\hbar}\exp(\mathrm{i}t)\right).$$

With the change of variable $s=(zr/\hbar)\exp(\mathrm{i}t)$ the integral is seen to vanish so (31) is fulfilled. The closed path around $w_0=\infty$ can be defined by $w(t)=(1/r)\exp(\mathrm{i}t)$, $r>0$, $-\pi\leqslant t<\pi$, in such a way that under the inversion $v=1/w$ the image path is a closed curve around $v=0$ in the clockwise sense, so

$$z\varphi_0(z)=\hbar\int_{-\pi}^{\pi}\mathrm{d}t\left[\frac{\mathrm{i}z}{\hbar r}\exp(\mathrm{i}t)\right]\exp\left(\frac{\mathrm{i}z}{\hbar r}\exp(\mathrm{i}t)\right).$$

Again the integral is a total differential and (31) is fulfilled.

For any of the above paths the value of $\varphi_0(z)$ is zero for $z\in\mathscr{C}$, so we have selected this particular value for the solution for $z=0$. When $z=\infty$ a path around the origin leads $\varphi_0(z)$ to the same value. Therefore (32) is defined for all values of z.

Since we choose $\hat{\Omega}_0=\hat{q}$, its image operator under a p-point CT is precisely the image position operator whose corresponding symbol is given by (22), i.e.

$$\Omega_1(z,v)=\frac{1}{g'(v)}\{z-\mathrm{i}\hbar\partial_v\ln B(g(v))\} \tag{33}$$

which yields the following expression for the intermediary differential equation (29):

$$\left\{\frac{1}{g'(v)}\{z'-\mathrm{i}\hbar\partial_\alpha\ln B(g(\alpha))\}|_{\alpha=g^{-1}(-\mathrm{i}\hbar\partial_z)}\right\}\varphi_1(z)|_{z'=z}=0. \tag{34}$$

The coefficients of this differential equation are necessarily linear in z because p-point CTs are linear in $\hat{q}$. If we assume that (34) is a differential equation of finite order we conclude that $1/g'(v)$ is a polynomial whose grade cannot exceed the order of the differential equation, the same being valid for $(1/g'(v))\partial_v\ln B(g(v))$. However, $B(g(v))$ may have poles of higher order or other kind of singularities.

A solution of (34) is then obtained by replacement of (32) in (26):

$$\varphi_1(z)=\oint_\gamma \mathrm{d}wB(w)\exp\left\{\frac{\mathrm{i}}{\hbar}zg^{-1}(w)\right\}.$$

With $w=g(v)$, $\mathrm{d}w=g'(v)\,\mathrm{d}v$, we obtain

$$\varphi_1(z)=\oint_\gamma \mathrm{d}vg'(v)B(g(v))\exp\left(\frac{\mathrm{i}}{\hbar}zv\right). \tag{35}$$

This is the general expression for a solution of the intermediary equation if $B(g(v))$ is analytic and single-valued on γ. For non-trivial solutions the allowed closed paths of integration, which may lead to different integral representations, must contain one or more singularities of the integrand. Therefore the study of the allowed solutions turns into an investigation of the analycity of $B(g(v))$ and used of functional analysis technics.

Continuing the procedure, under a q-point CT the intermediary equation is mapped into the desired differential equation (30), whose solution is directly obtained by inserting (35) in (27):

$$\psi(z)=A(z)\oint_\gamma \mathrm{d}vg'(v)B(g(v))\exp\left(\frac{\mathrm{i}}{\hbar}f(z)v\right). \tag{36}$$

Since a q-point CT is linear in $\hat{p}$, the differential equations (30) and (34) must have the same order. Since their coefficients may be different, new singularities of (30) with respect to (34) have to be introduced by the functions f and A. For non-trivial solutions the allowed paths of integration in (36) are the same as in (35). However, if (30) is a time-independent Schrödinger equation the functions $f(z)$ and $A(z)$, as well as the singularities of $B(g(v))$, must yield bounded solutions in the square mean. In the examples we will see that the energy α is one of the parameters that define the kind of singularities of $B(g(v))$. In principle this method may yield states for the continuous and discrete spectra. One way to obtain discrete states is by selecting α so that the singularities of $B(g(v))$ become poles of order n and choosing a path of integration around one pole in a domain where the integrand in (36) is analytic. For continuous states we must select α real and study the possible paths which may lead to physical solutions.

We now explicitly describe how the functions $g'(v)$, $B(g(v))$, $A(z)$ and $f(z)$ may be obtained when (30) is a second-order differential equation. That is when $\Omega(z, v)$ is quadratic in v:

$$\Omega(z, v) = f_1(z)v^2 + f_2(z)v + f_3(z) \tag{37}$$

where $f_i(z)$, $i = 1, 2, 3$, are, in general, polynomials nonlinear in z. A similar procedure may be followed for other choices of $\hat{\Omega}$.

According to (22) and (23) the first transformation which maps $\hat{\Omega}_0$ into $\hat{\Omega}_1$ is characterized by the following mapping equations:

$$Q_1(z, v) = \frac{1}{g'(v)}\{z - i\hbar\partial_v \ln B(g(v))\}$$

$$P_1(z, v) = g(v).$$

Since $\hat{\Omega}_1 = \hat{Q}_1$, its normal symbol must be linear in z and may have the form

$$\begin{aligned}\Omega_1(z, v) &= (az + b)v^2 + (cz + d)v - \alpha_1 \\ &= h_1(z)v^2 + h_2(z)v - \alpha_1\end{aligned} \tag{38}$$

where a, b, c, d and α_1 are complex parameters. From (33) and (38) we obtain

$$\frac{1}{g'(v)}\{z - i\hbar\partial_v \ln B(g(v))\} = (av + c)vz + v(bv + d) - \alpha_1$$

so

$$\begin{aligned}g'(v) &= \frac{1}{v(av + c)} \\ \partial_v \ln B(g(v)) &= \frac{i}{\hbar}\frac{v(bv + d) - \alpha_1}{v(av + c)}.\end{aligned} \tag{39}$$

Integration of the last equation leads to

$$B(g(v)) = C\exp\left\{\frac{ib}{\hbar a}v + \frac{ibc}{\hbar a^2} + \frac{i}{\hbar}\left(\frac{d}{a} + \frac{\alpha_1}{c} - \frac{bc}{a^2}\right)\ln(av + c) - \frac{i}{\hbar}\frac{\alpha_1}{c}\ln v\right\} \tag{40}$$

where C is a complex constant and $a \neq 0$, $c \neq 0$. When $a = 0$ we obtain

$$g'(v) = \frac{1}{cv} \tag{41}$$

$$B(g(v)) = C\exp\left\{\frac{i}{2\hbar}\frac{b}{c}v^2 + \frac{i}{\hbar}\frac{d}{c}v - \frac{i}{\hbar}\frac{\alpha_1}{c}\ln v\right\}. \tag{42}$$

These equations show that the order and position of the poles in $g'(v)$, as well as the singularities of $B(g(v))$, depend on the value of the five parameters a, b, c, d and α_1, which at the same time are defining the coefficients of the intermediary differential equation.

The second transformations maps $\hat{\Omega}_1$ into the desired $\hat{\Omega}$ whose normal symbol in the case of second-order differential equations is given by (37). Recalling (17) and (18) the corresponding mapping equations are

$$P_2(z,v)=\frac{1}{f'(z)}\{v+i\hbar\partial_z\ln A(z)\}.$$
$$Q_2(z,v)=f(z).$$

To obtain the symbol of $\hat{\Omega}$ in terms of $Q_2(z,v)$ and $P_2(z,v)$ we first determine $\hat{\Omega}$ in terms of $\hat{Q}_2$ and $\hat{P}_2$. Since $\hat{\Omega}_1=h_1(\hat{q})\hat{p}^2+h_2(\hat{q})\hat{p}-\alpha_1$ and

$$\hat{T}\hat{\Omega}_1=\{h_1(\hat{Q}_2)\hat{P}_2\hat{P}_2+h_2(\hat{Q}_2)\hat{P}_2-\alpha_1\}\hat{T}=\hat{\Omega}\hat{T}$$

we obtain

$$\hat{\Omega}=h_1(\hat{Q}_2)\hat{P}_2\hat{P}_2+h_2(\hat{Q}_2)\hat{P}_2-\alpha_1.$$

Here we have used $\hat{T}\hat{q}=\hat{Q}_2\hat{T}$ and $\hat{T}\hat{p}=\hat{P}_2\hat{T}$. Applying (7) together with the mapping equations it follows:

$$\Omega(z,v)=h_1(f(z))(P_2\circ P_2)(z,v)+h_2(f(z))P_2(z,v)-\alpha_1$$

where

$$(P_2\circ P_2)(z,v)=(P_2(z,v))^2-i\hbar\frac{1}{f'(z)}\partial_z P_2(z,v)$$

and hence

$$\Omega(z,v)=\Omega_1(Q_2(z,v),P_2(z,v))-i\hbar\frac{h_1(f(z))}{f'(z)}\partial_z P_2(z,v). \tag{43}$$

Therefore the exact symbol of $\hat{\Omega}$ differs from the classical expression $\Omega(z,v)=\Omega_1(Q_2(z,v),P_2(z,v))$ in the second term on the right-hand side of (43). Equating (37) and (43) one obtains the equation from which $f(z)$ and $A(z)$ can be determined.

We note that this method allows one to work with canonical transformations to construct exact solutions by applying the standard methods of classical mechanics under the condition that the normal symbol $\Omega(z,v)$ in terms of $Q_2(z,v)$ and $P_2(z,v)$ represents exactly $\hat{\Omega}$, for which (43) must be fulfilled in the case of second-order differential equations.

6. Examples

We now present some standard examples for second-order differential equations to show how this method works and that (36) yields known closed contour integral representations for the solution.

6.1. Kummer's differential equation

Let us consider Kummer's differential equation which has a wide range of applications in physics [12, 13]:

$$\Omega(z',-i\hbar\partial_z)\psi(z)|_{z'=z}=\{z(-i\hbar\partial_z)^2+(\beta-z)(-i\hbar\partial_z)-\alpha_1\}\psi(z)=0$$

where α_1 and β are complex parameters. The corresponding symbol for $\hat{\Omega}$ is

$$\Omega(z,v)=zv^2+(\beta-z)v-\alpha_1. \tag{44}$$

Since the coefficients are linear in z only a p-point CT is required to map $\hat{\Omega}_0 = \hat{q}$ into $\hat{\Omega}$. In this case $\psi(z)$ is given by (35) with $\varphi_1(z) = \psi(z)$ or by (36) with $f(z) = z$ and $A(z) = 1$ which correspond to the identity transformation for the second step. That is

$$\psi(z) = \oint_\gamma \mathrm{d}v g'(v) B(g(v)) \exp\left(\frac{\mathrm{i}}{\hbar} vz\right).$$

Since equation (38) with $a = 1$, $b = 0$, $c = -1$ and $d = \beta$ yields (44), from (39) and (40) we obtain

$$g'(v) = \frac{1}{v(v-1)} \qquad B(g(v)) = C(v-1)^{i/\hbar(\beta - \alpha_1)}(v)^{(i/\hbar)\alpha_1}.$$

So

$$\psi_{\beta,\alpha_1}(z) = C \oint_\gamma \mathrm{d}v (v-1)^{(i/\hbar)\beta - (i/\hbar)\alpha_1 - 1}(v)^{(i/\hbar)\alpha_1 - 1} \exp\left(\frac{\mathrm{i}}{\hbar} vz\right). \tag{45}$$

This equation gives the closed contour integral representation of Kummer's differential equation up to the constant factor C. For general values of β and α_1 the function $B(g(v))$ is analytic on the complex plane except on the branch cuts of $\ln(v-1)$ and $\ln(v)$. Selecting these branch cuts on the real axis from 1 to ∞ and from 0 to $-\infty$, respectively, a path of integration going counterclockwise around the branch points $v = 1$ and $v = 0$ and then clockwise around the same points, the equation (45) for $\Re(\mathrm{i}\beta/\hbar) > \Re(\mathrm{i}\alpha_1/\hbar) > 0$ yields

$$\psi_{\beta,\alpha_1}(z) = D_1 \int_0^1 \mathrm{d}y (y-1)^{(i/\hbar)\beta - (i/\hbar)\alpha_1 - 1}(y)^{(i/\hbar)\alpha_1 - 1} \exp\left(\frac{\mathrm{i}}{\hbar} yz\right). \tag{46}$$

where D_1 is a constant factor depending on the parameters β and α_1. This equation is the known real integral representation of the confluent hypergeometric function (see equation (13) Chapter 1 in [12]).

6.2. Whittaker's differential equation

Let us now study Whittaker's differential equation:

$$\left\{ z(-\mathrm{i}\hbar\partial_z)^2 - \beta/2(\beta/2 + \mathrm{i}\hbar)\frac{1}{z} - \frac{1}{4}z + \beta/2 - \alpha \right\}\psi(z) = 0.$$

The corresponding normal symbol for the pseudodifferential operator is

$$\Omega(z, v) = zv^2 - \beta/2(\beta/2 + \mathrm{i}\hbar)\frac{1}{z} - \frac{1}{4}z + \beta/2 - \alpha. \tag{47}$$

Whittaker's differential equation is the image of Kummer's equation under the

q-point CT which eliminates the linear term in ∂_z as we show below.

Taking Kummer's equation as the intermediary equation we have

$$\Omega_1(z, v) = zv^2 + (\beta - z)v - \alpha_1.$$

With $f(z) = z$ the mapping equations for the second step are

$$Q_2(z, v) = f(z) \qquad P_2(z, v) = v + \mathrm{i}\hbar\partial_z \ln A(z)$$

and (43) leads to

$$\Omega(z, v) = zv^2 + (2\mathrm{i}\hbar z\partial_z \ln A(z) + \beta - z)v + [-\hbar^2 z(\partial_z \ln A(z))^2 + \hbar^2 z\partial_z^2 \ln A(z) + (\beta - z)\,\mathrm{i}\hbar\partial_z \ln A(z)] - \alpha_1.$$

Now choosing $A(z)$ so that the linear term in v vanishes we obtain

$$A(z) = C\exp\left(-\frac{\mathrm{i}z}{2\hbar}\right)(z)^{(\mathrm{i}/\hbar)(\beta/2)}$$

where C is a constant factor. With this choice and $\alpha_1 = \alpha$ the intermediary symbol is mapped into (47), so that with $\varphi_1(z)$ given by (45) the equation (36) yields

$$\psi(z) = D'\exp\left(-\frac{\mathrm{i}z}{2\hbar}\right)(z)^{(\mathrm{i}/\hbar)(\beta/2)}\oint_\gamma \mathrm{d}v(v-1)^{(\mathrm{i}/\hbar)\beta-(\mathrm{i}/\hbar)\alpha-1}(v)^{(\mathrm{i}/\hbar)\alpha-1}\exp\left(\frac{\mathrm{i}}{\hbar}vz\right)$$

which is the closed contour integral representation for a solution of Whittaker's equation up to the constant factor D'. For $\Re(\mathrm{i}\beta/\hbar) > \Re(\mathrm{i}\alpha/\hbar) > 0$ and the same branches and path of integration given for (45), the last equation yields the known real integral representation for the solution (see equation 3a in chapter 2 of [12]):

$$\psi(z) = N\exp\left(-\frac{\mathrm{i}z}{2\hbar}\right)(z)^{(\mathrm{i}/\hbar)(\beta/2)}\psi_{\beta,\alpha}(z)$$

where N is a constant factor and $\psi_{\beta,\alpha}$ is given by (46).

6.3. *The harmonic oscillator*

A standard differential equation in physics is that of the harmonic oscillator:

$$\Omega(z', -\mathrm{i}\hbar\partial_z)\psi(z)|_{z'=z} = \left\{-\frac{\hbar^2}{2m}\partial_z^2 + \frac{m\omega^2}{2}z^2 - \alpha\right\}\psi(z) = 0$$

for which

$$\Omega(z, v) = H(z, v) - \alpha = \frac{1}{2m}v^2 + \frac{m\omega^2}{2}z^2 - \alpha \tag{48}$$

where $H(z, v)$ is the complex symbol of the Hamiltonian operator for a particle of mass m describing a simple harmonic movement with angular frequency $\omega \in \Re$.

Under the p-point CT characterized by

$$g'(v) = \frac{1}{cv} \qquad B(g(v)) = D\exp\left(\frac{ib}{\hbar c}v^2 - \frac{\mathrm{i}\alpha_1}{\hbar c}\ln(v)\right)$$

$\Omega_0(z, v)$ is mapped into the intermediary equation $\Omega_1(z, v) = bv^2 + czv - \alpha_1$. Under the q-point CT defined by

$$f(z) = az \qquad A(z) = C\exp\left(\frac{i}{\hbar}\frac{ca^2}{4b}z^2\right)$$

where C is a constant, theintermediary equation is mapped into

$$\Omega(z, v) = \frac{b}{a^2}v^2 - \frac{c^2a^2}{4b}z^2 - \left(\alpha_1 - \frac{i\hbar c}{2}\right).$$

With $a^2 = 4i\hbar mc$, $b = 2i\hbar c$, $c = \pm i\omega$ and $\alpha_1 = \alpha + \frac{1}{2}i\hbar c$, this equation yields the desired symbol (48). From (36) one infers that for physical solutions $A(z)$ must converge for $\Re z \to \infty$ for which we select the positive value of c. Hence (36) for z real leads to

$$\psi(x) = N\exp(-\tfrac{1}{2}\beta^2x^2)\oint_\gamma \mathrm{d}v(v)^{-\alpha/(\hbar\omega)-1/2}\exp(-v^2)\exp(2\beta xv) \qquad (49)$$

where $\beta = -(m\omega/\hbar)^{1/2}$ and N is a constant factor. For general values of α, the function $B(g(v))$ and the integrand are analytic on the complex plane except on the branch cut of $\ln(v)$. To obtain discrete states $(-\alpha/(\hbar\omega) + \frac{1}{2})$ must be a negative integer, because for these values of α the branch point $v = 0$ of $B(g(v))$ becomes a pole of order n. With the path of integration encircling the origin the integral in (49) leads to the known closed contour integral representation for the Hermite polynomials $H_n(\beta x)$ (see equation 22.10.9 in [14]) and hence

$$\psi_n(x) = N\exp(-\tfrac{1}{2}\beta^2x^2)H_n(\beta x)$$

gives, up to the constant factor N, the familiar form for the eigenstates of the harmonic oscillator corresponding to the eigenvalues $\alpha = \hbar\omega(n + \frac{1}{2})$ (see equation (30.19) in [15]).

7. Conclusions

The complex extension for the point transformations as well as for the solution were obtained in sections 3 and 4 thanks to Krüger's definition for ordered pseudodifferential operators, which permits one to avoid the use of integral representations like Fourier transformations and delta Dirac distributions.

We have shown that a closed contour integral representation for an exact solution of one-dimensional linear differential equations can be obtained by canonical methods mapping the simple equation (31) into the desired differential equation under two successive complex-valued point transformations on the Cartesian momentum and position spaces. This method may lead to new integral representations and has the advantage that the transformations can be worked out in a classical way mapping z into the exact symbol of the desired image pseudodifferential operator. When this method is applied to solve Schrödinger equations the unitarity condition on the generator is not required and hence the problem of the completeness of the intermediary states is avoided.

For future researches it should be interesting to study higher sequences of these point transformations starting with the same equation (31) to obtain exact solutions or uniform approximations of a wide range of differential equations. To obtain this, the crucial point is the exact calculation of (26) for a third step or a formal determination of an approximation for this equation, for example up to $\mathcal{O}(\hbar^2)$. Other possibilities might consider canonical transformations, different from point transformations, for which a similar study to that carried out in section 3 should be made.

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